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Some characterizations of self-adjoint operators

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Let H be a separable, infinite dimensional complex Hilbert space, and let $B(H)$ denote the algebra of all bounded linear operators on H. For $T\in B(H)$, the absolute part of T, denoted by $|T|$, is defined as usual as the positive square root of T^*T. Each $T\in B(H)$ can be uniquely expressed as $A+iB$, where A, B are self-adjoint operators called the real part and the imaginary part respectively, denoted by $\mathrm{Re}\,T$ and $\mathrm{Im}\,T$, respectively. Note that $\mathrm{Re}\,T=(T+T^*)/2$ and $\mathrm{Im}\,T=(T-T^*)/2i$.

The following two theorems are characterization of self-adjoint and positive operators and were obtained by FONG and ISTRATESCU [1] and FONG and TSUI [2], respectively.

Theorem A. *An operator $T\in B(H)$ is self-adjoint if and only if $|T|^2\leqq(\mathrm{Re}\,T)^2$.*

Theorem B. *An operator $T\in B(H)$ is positive if and only if $|T|\leqq\mathrm{Re}\,T$.*

The purpose of this note is to generalize Theorem A as well as to present a new proof of Theorem B which may lead to further development in this direction. At the end of this paper we will give some characterization modulo C_p (the Schatten p-class) of self-adjoint operators.

Recall that $T\in B(H)$ is said to be hyponormal if $TT^*\leqq T^*T$ and in this case the spectral radius $r(T)=\|T\|$ (see [6]).

Theorem 1. *Let $T\in B(H)$ be hyponormal. If for some $S\in B(H)$ and a complex number α, $|T|^2+\alpha(TS-ST)\leqq 0$, then $T=0$.*

Proof. Since $r(T)=\|T\|$, there exists a sequence $\{x_n\}$ of unit vectors in H such that $(T-t)x_n\to 0$ where $|t|=\|T\|$. Now $(T^*Tx_n, x_n)+\alpha(TSx_n, x_n)-\alpha(STx_n, x_n)\leqq$ $\leqq 0$. Hence $\|Tx_n\|^2+\alpha(Sx_n, (T-t)^*x_n)-\alpha((T-t)x_n, S^*x_n)\leqq 0$. But since T is hyponormal and $\|(T-t)x_n\|\to 0$, it follows that $\|(T-t)^*x_n\|\to 0$. Letting $n\to\infty$, in the last inequality, we obtain $|t|^2\leqq 0$. Hence $T=0$ as required.

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Corollary 1. *If* $|T|^2 \leqq (\mathrm{Re}\,T)^2$, *then* $T = T^*$.

Proof. Let $T = A + iB$. Then $|T|^2 \leqq (\mathrm{Re}\,T)^2$ is equivalent to $B^2 + i(AB - BA) \leqq \leqq 0$. Now the corollary follows from Theorem 1.

The following proof of Theorem B was suggested to me by J. Stampfli.

Lemma 1. *Let* $T \in B(H)$ *be such that* $T = VP$ *where* V *is a contraction,* $P \geqq 0$ *and* $2P \leqq VP + PV^*$. *Let* $P = D + K$ *where* D *is diagonal and positive and* K *is arbitrary. If* $Dx = \lambda x$ *with* x *a unit vector in* H *and* $\lambda > 0$, *then* $\|(1-V)^*x\| \leqq \leqq (2/\lambda)\|Kx\|$.

Proof. Observe first that $\|(1-V)^*x\|^2 \leqq 2 - ((V+V^*)x, x)$. Now

$$\begin{aligned} 2\lambda + 2(Kx, x) &= 2(Px, x) \leqq (VPx, x) + (PV^*x, x) = \\ &= (VDx, x) + (DV^*x, x) + (VKx, x) + (KV^*x, x) = \\ &= \lambda((V+V^*)x, x) + (VKx, x) + (KV^*x, x). \end{aligned}$$

Therefore $\lambda[2 - ((V+V^*)x, x)] \leqq ((V-1)Kx, x) + (K(V^*-1)x, x)$ and so

$$2 - ((V+V^*)x, x) \leqq (2/\lambda)\,\|(V-1)^*x\|\,\|Kx\|.$$

Combining this inequality with the first inequality, we obtain

$$\|(1-V)^*x\| \leqq (2/\lambda)\,\|Kx\|$$

as required.

An alternative proof of Theorem B. Let $T = VP$ be the polar decomposition of T. Let $P = \int t\,dE(t)$ where $E(t)$ is the spectral measure of P. Fix $\alpha > 0$ and let $H_\alpha = E([\alpha, \infty))H$. If $\varepsilon > 0$ is given, then by Weyl—Von Neumann Theorem [3], $P_\alpha = D + K$ where D is diagonal and K is Hilbert—Schmidt with $\|K\|_2 < \varepsilon$ ($\|\cdot\|_2$ denotes the Hilbert—Schmidt norm). If $De_n = \lambda_n e_n$ where $\{e_n\}$ is a basis for H_α, then for any unit vector $y \in H_\alpha$, $y = \sum_{n=1}^{\infty} a_n e_n$ for some a_n with $\sum_{n=1}^{\infty} |a_n|^2 = 1$. Applying the lemma, we obtain

$$\begin{aligned} \|(1-V)^*y\| &\leqq \sum_{n=1}^{\infty} \|(1-V)^* a_n e_n\| \leqq \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} \|(1-V)^* e_n\|^2\right)^{1/2} \leqq \\ &\leqq (2/\alpha)\left(\sum_{n=1}^{\infty} \|Ke_n\|^2\right)^{1/2} < (2/\alpha)\varepsilon. \end{aligned}$$

Since ε is arbitrary, $V = 1$ on H_α. Since $\alpha > 0$ is arbitrary we have $V = 1$ on $(\ker P)^\perp = = \overline{R(P)}$. Therefore $T = VP = P \geqq 0$ as required.

We remark that the above proof works for the following generalization of Theorem B.

Theorem 2. *If* $P \geqq 0$, V *is a contraction and* $2P \leqq VP + PV^*$, *then* $P = VP$ *and* $V|_{(\mathrm{Ker}\, P)^\perp} = 1$.

In what follows we shall prove that if $|T|^2 - (\mathrm{Re}\, T)^2 \in C_p$ $(p \geqq 1)$, then $T - T^* \in C_{2p}$. Recall that a compact operator C is in C_p if and only if $\|C\|_p^p = \sum_{i=1}^{\infty} s_i(C)^p < \infty$ where $s_1(C), s_2(C), \dots$ denotes the sequence of eigenvalues of $|C|$ in decreasing order and repeated according to multiplicity. It is known (see [7]) that for $p \geqq 1$, $\|C\|_p^p \geqq \sum_{n=1}^{\infty} |(Ce_n, f_n)|^p$ for any orthonormal sets $\{e_n\}$ and $\{f_n\}$ in H. We refer to [5] or [7] for further properties of the Schatten p-classes.

Lemma 2. *Let* $T \in B(H)$ *be hyponormal. If for some* $S \in B(H)$ *and a complex number* α, $|T|^2 + \alpha(TS - ST)$ *is compact, then* T *is compact.*

Proof. Let $K(H)$ denote the closed ideal of compact operators in $B(H)$, and let $\pi: B(H) \to B(H)/K(H)$ be the quotient map of $B(H)$ onto the Calkin algebra $B(H)/K(H)$. Therefore $|\pi(T)|^2 + \alpha(\pi(T)\pi(S) - \pi(S)\pi(T)) = 0$ and so by Theorem 1 we have $\pi(T) = 0$, in other words, T is compact. (Recall that the Calkin algebra is a B^*-algebra and so it is representable as an operator algebra.)

Theorem 3. *Let* $T \in B(H)$ *be hyponormal. If for some* $S \in B(H)$ *and a complex number* α, $|T|^2 + \alpha(TS - ST) \in C_p$ $(p \geqq 1)$, *then* $T \in C_{2p}$.

Proof. Since $C_p \subset K(H)$, we have by Lemma 2 that $T \in K(H)$. But it is known [6] that a compact hyponormal operator is diagonal, therefore $Te_n = \lambda_n e_n$ for some basis $\{e_n\}$ of H. Thus

$$\infty > \| |T|^2 + \alpha(TS - ST) \|_p^p \geqq \sum_{n=1}^{\infty} \left| \left(|T|^2 + \alpha(TS - ST) e_n, e_n \right) \right|^p =$$

$$= \sum_{n=1}^{\infty} \left| \|Te_n\|^2 + \alpha(Se_n, T^* e_n) - \alpha(Te_n, S^* e_n) \right|^p =$$

$$= \sum_{n=1}^{\infty} \left| |\lambda_n|^2 + \alpha\lambda_n (Se_n, e_n) - \alpha\lambda_n (e_n, S^* e_n) \right|^p = \sum_{n=1}^{\infty} |\lambda_n|^{2p}$$

and so $T \in C_{2p}$ as required.

Corollary. *If* $|T|^2 - (\mathrm{Re} T)^2 \in C_p$ $(p \geqq 1)$, *then* $T - T^* \in C_{2p}$. *Hence* T *has a non-trivial invariant subspace.*

Proof. Observe that $|T|^2 - (\mathrm{Re} T)^2 = B^2 + i(AB - BA) \in C_p$ and apply Theorem 3 to get $B \in C_{2p}$. The last assertion follows from Corollary 6.15 in [4] (which says that if $T - T^* \in C_p$ for some $p \geqq 1$, then T has a non-trivial invariant subspace).

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